1. (+) Let $G = \mathbb{R}[x]$ denote the group of all polynomials with real coefficients under addition.
   (a) For any $f \in \mathbb{R}[x]$, define $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ by $\phi(f) = f'$, the derivative of f. Show that ϕ defines a homomorphism.
   (b) Let $\int f$ denote the antiderivative of f through the point $(0, 0)$. Show that the mapping $f \mapsto \int f$ from G to G defines a homomorphism. What is the kernel of this mapping?
   (c) Is the map in (b) still a homomorphism if $\int f$ denotes the antiderivative of f through the point $(0, 1)$?

2. (+) Prove that the mapping $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $(a, b) \mapsto a - b$ is a homomorphism. What is the kernel of ϕ? Describe the set $\phi^{-1}(3)$ (that is, all elements that map to 3).

3. In the following, compute the indicated quantities for the given homomorphism.
   (a) $\text{Ker}(\phi)$ and $\phi(18)$ for $\phi : \mathbb{Z} \to S_8$ such that $\phi(1) = (1426)(257)$.
   (b) $\text{Ker}(\phi)$ and $\phi(-3, 2)$ for $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where $\phi(1, 0) = 3$ and $\phi(0, 1) = -4$
   (c) (+) $\text{Ker}(\phi)$ and $\phi(15)$ for $\phi : \mathbb{Z}_{24} \to S_8$ where $\phi(1) = (25)(1467)$.

4. Suppose that k is a divisor of n. Prove that $\mathbb{Z}_n/\langle k\rangle \simeq \mathbb{Z}_k$.

5. (+) Suppose that $\phi : \mathbb{Z}^*_{40} \to \mathbb{Z}^*_{40}$ is a homomorphism. (Recall, $\mathbb{Z}^*_{40}$ is the group of all elements of $\mathbb{Z}_{40}$ that have a multiplicative inverse with operation $\cdot$ modulo 40.) If $\text{Ker}\phi = \{1, 9, 7, 17, 33\}$ and $\phi(11) = 11$, find all elements of $\mathbb{Z}^*_{40}$ that map to 11.

6. For the following, give an example of a nontrivial homomorphism ϕ for the given groups, if an example exists. If no such homomorphism exists, explain why that is so.
   (a) (+) $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_5$ (b) $\phi : \mathbb{Z} \to S_3$ (c) $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_4$

7. Let $\phi : G \to G'$ be a homomorphism with kernel H and let $a \in G$. Prove the set equality $\{x \in G \mid \phi(x) = \phi(a)\} = Ha$.

8. (*) Let G be a group, $h \in G$ and $n \in \mathbb{Z}^+$. Let $\phi : \mathbb{Z}_n \to G$ be defined by $\phi(i) = h^i$ for $0 \le i \le n$. Give a necessary and sufficient condition (in terms of h and n) for ϕ to be a homomorphism. Prove your assertion.

9. (+) Let G be a group that is generated by the set $\{a_i \mid i \in I\}$ where I is some indexing set and $a_i \in G$ for all $i \in I$. Let $\phi : G \to G'$ and $\mu : G \to G'$ be two homomorphisms from G into a group G' such that $\phi(a_i) = \mu(a_i)$ for all $i \in I$. Prove that $\phi = \mu$. (This shows that any homomorphism of G is determined by its action on the generators of G.)

10. How many homomorphisms are there of $\mathbb{Z}$ into $\mathbb{Z}_2$?