1. (+) Determine if the set $\{a+b\sqrt{2} \mid a,b \in \mathbb{Z}\}$ with the usual addition and multiplication defines a ring. If a ring is not formed, explain why. If a ring is formed, determine whether it has unity and whether it is a field.

2. Describe all units in the given ring

   (a) (+) $\mathbb{Z}_6$ (b) $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Q}$

3. (+) Consider the map det of $M_n(\mathbb{R})$ onto $\mathbb{R}$ where $\det(A)$ is the determinant of the matrix A. Is det a ring homomorphism? Why or why not?

4. (+) Consider this solution of the equation $X^2 = I_3$ in the ring $M_3(\mathbb{R})$.

   $X^2 = I_3$ implies $X^2 - I_3 = 0$, so factoring, we have $(X - I_3)(X + I_3) = 0$, whence either $X = I_3$ or $X = -I_3$.

   Is this reasoning correct? If not, point out the error, and, if possible, give a counterexample.

5. Find all solutions of $x^2 + x - 6 = 0$ in the ring (a) $\mathbb{Z}_{14}$; (b) $\mathbb{Z}_{16}$

6. (+) Find all solutions of the equation $x^2 + 2x + 4 = 0$ in $\mathbb{Z}_6$.

7. Find the characteristic of the given ring.

   (a) (+) $4\mathbb{Z}$ (b) $\mathbb{Z}_5 \times \mathbb{Z}_3$

8. List the zero divisors in $\mathbb{Z}_9$.

9. Show by example that for fixed nonzero elements a and b in a ring, the equation $ax = b$ can have more than one solution. How does this compare with groups?

10. (+) Let p be a prime. Show that in the ring $\mathbb{Z}_p$ we have $(a+b)^p = a^p + b^p$ for all $a, b \in \mathbb{Z}_p$.

11. (+) An element a in a ring R is **idempotent** if $a^2 = a$.

    (a) Show that the set of all idempotent elements of a commutative ring is closed under multiplication.

    (b) Find all idempotents in the ring $\mathbb{Z}_4 \times \mathbb{Z}_{12}$.

12. Show that a division ring contains exactly two idempotent elements.

13. (*) Let n be an integer greater than 1. In a ring in which $x^n = x$ for all x, show that $ab = 0$ implies $ba = 0$.

14. (*) Show that the characteristic of an integral domain must be either 0 or prime.