Those problems marked with (+) will be collected in class for a grade. I will announce the due date one class period in advance. Those problems marked with (∗) denote problems that only those students taking this course as an honors course must complete. The remaining problems may be discussed in class and students should post solutions on the wiki.

1. Determine whether the given maps ϕ determine an isomorphism of the first binary structure with the second. If it isn't an isomorphism, why not?
   (a) $\langle \mathbb{Z}, + \rangle$ with $\langle \mathbb{Z}, + \rangle$ where $\phi(n) = 3n + 4$.
   (b) (+) $\langle \mathbb{R}, + \rangle$ with $\langle \mathbb{R}^+, \cdot \rangle$ where $\phi(r) = 0.5^r$ for $r \in \mathbb{R}$.
   (c) (+) $\langle F, + \rangle$ with $\langle F, + \rangle$ where F is the set of all functions $f : \mathbb{R} \to \mathbb{R}$ that have derivatives of all orders and $\phi(f) = f'$, the derivative of f.
2. (+) The map $\phi : \mathbb{Z} \to \mathbb{Z}$ defined by $\phi(n) = n + 1$ is one-to-one and onto $\mathbb{Z}$. Give the definition of a binary operation $*$ on $\mathbb{Z}$ such that ϕ is an isomorphism mapping
   (a) $\langle \mathbb{Z}, + \rangle$ onto $\langle \mathbb{Z}, * \rangle$
   (b) $\langle \mathbb{Z}, * \rangle$ onto $\langle \mathbb{Z}, + \rangle$

   In each case, give the identity element of $\langle \mathbb{Z}, * \rangle$.
3. (+) Read the explanation about a proof synopsis on page 35, then write a proof synopsis of Theorem 3.13.
4. (*) An identity element for a binary operation $*$ as described in class is sometimes called a "two-sided identity element". Using complete sentences, give analogous definitions for
   (a) a left identity element e_L for $*$, and
   (b) a right identity element e_R for $*$

   Theorem 3.13 shows that if a two-sided identity element for $*$ exists, it is unique. Is the same true for a one-sided identity element you just defined? If so, prove it. If not, give a counter example $\langle S, * \rangle$ for a finite set S and find the first place where the proof of Theorem 3.13 breaks down.
5. (+) Give a careful proof for a skeptic that the property "For each $c \in S$, the equation $x * x = c$ has a solution x in S." is a structural property. (This is done for the property "There is an identity element for $*$" in Theorem 3.14.)
6. Recall that if $f : A \to B$ is a one-to-one function mapping A onto B, then $f^{-1}(b)$ is the unique $a \in A$ such that $f(a) = b$. Prove that if $\phi : S \to S'$ is an isomorphism of $\langle S, * \rangle$ with $\langle S', *' \rangle$, then ϕ^{-1} is an isomorphism of $\langle S', *' \rangle$ with $\langle S, * \rangle$.
7. Prove that if $\phi : S \to S'$ is an isomorphism of $\langle S, * \rangle$ with $\langle S', *' \rangle$ and $\psi : S' \to S''$ is an isomorphism of $\langle S', *' \rangle$ with $\langle S'', *'' \rangle$, then the composite function $\psi \circ \phi$ is an isomorphism of $\langle S, * \rangle$ with $\langle S'', *'' \rangle$.
8. (+) Prove that the relation $\simeq$ of being isomorphic is an equivalence relation on any set of binary structures. You may simply quote the results you were asked to prove in the preceding two exercises at the appropriate places in your proof.
9. Let H be the subset of $M_2(\mathbb{R})$ consisting of all matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for $a, b \in \mathbb{R}$. One can show H is closed under matrix addition and matrix multiplication.
   (a) Show that $\langle \mathbb{C}, + \rangle \simeq \langle H, + \rangle$.
   (b) Show that $\langle \mathbb{C}, \cdot \rangle \simeq \langle H, \cdot \rangle$.