Those problems marked with (+) will be collected in class for a grade. I will announce the due date one class period in advance. Those problems marked with ($*$) denote problems that only those students taking this course as an honors course must complete. The remaining problems may be discussed in class and students should post solutions on the wiki.

1. (a) Show that multiplication is a well-defined binary operation on the set $\mathbb{Z}_n$ of congruence classes of integers modulo n.
   (b) (+) Given an integer $n \geq 1$, let $\mathbb{Z}_n^*$ be the set of elements $\bar{x} \in \mathbb{Z}_n$ such that there exists $\bar{y} \in \mathbb{Z}_n$ with $\overline{xy} = \bar{1}$. Show that $\mathbb{Z}_n^*$ with the operation of multiplication is a group.
   (c) Write multiplication tables for $\mathbb{Z}_8^*$, $\mathbb{Z}_{10}^*$ and $\mathbb{Z}_{12}^*$.
   (d) (+) Show that $\mathbb{Z}_8^* \simeq \mathbb{Z}_{12}^*$, but that $\mathbb{Z}_{10}^*$ is not isomorphic to $\mathbb{Z}_8^*$ and $\mathbb{Z}_{12}^*$

2. Give an example of a group which has exactly 231 elements.

3. (+) In each case below find the inverse of the element under the given operation.

   (a) 13 in $\langle \mathbb{Z}_{20}, +_{20} \rangle$
   (b) ζ^{13} in $\langle U_{13}, \cdot \rangle$
   (c) $3 - 2i$ in $\mathbb{C}^*$, the group of nonzero complex numbers under multiplication

4. For a fixed point $(a, b) \in \mathbb{R}^2$, define $T_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ by $T_{a,b}(x, y) = (a + x, b + y)$. Show that $G = \{T_{a,b} \mid a, b \in \mathbb{R}\}$ is a group under function composition, $\circ$.

5. (+) Prove that the set of all rational numbers of the form $3^m 6^n$, where $m, n \in \mathbb{Z}$ is a group under multiplication.

6. (+) Let G be a group and let $g \in G$ be fixed. Show that the map $\gamma_g : G \to G$ defined by $\gamma_g(x) = gxg'$ is an isomorphism of G with itself.

7. Let G be a group with a finite number of elements. For $a \in G$, we define $a^n = a * a * \cdots * a$ for n factors. Show that for any $a \in G$, there exists an $n \in \mathbb{Z}^+$ such that $a^n = e$. [Hint: Consider $e, a, a^2, a^3, \ldots, a^m$ where m is the number of elements in G, and use the cancellation laws.]

8. Fraleigh, Section 4, number 20

9. (*) The following problem shows that we can loosen the requirements in the last two group axioms and still get a group. Consider a set G with a binary operation $*$ such that

   - $*$ is associative
   - there exists a *left identity element* $e \in G$ such that $e * x = x$ for all $x \in G$
   - for each $a \in G$ there exists a *left inverse* $a' \in G$ such that $a' * a = e$.

   (a) Show that left cancellation holds. That is, if $a * b = a * c$, then $b = c$.
   (b) Show that the left identity element e is also a right identity element for all $x \in G$.
   (c) Show that the left inverse a' for a is also a right inverse for a.

10. (*) Let p and q be distinct primes. Suppose that H is a proper subset of the integers that is a group under addition that contains exactly three elements of the set $\{p, p + q, pq, p^q, q^p\}$. Determine which of the following are the three elements in H.

    (a) pq, p^q, q^p
    (b) $p + q, pq, p^q$
    (c) p, pq, p^q
    (d) $p, p + q, pq$
    (e) p, p^q, q^p