Read each question carefully. Be sure to answer completely and show all of your work.

1. The following questions involve elements of S_6.

   (a) Write $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 4 & 2 & 1 \end{pmatrix}$ as the product of disjoint cycles.
   Solution: (136)(25)

   (b) Write the permutation (2634)(124) as the product of disjoint cycles.
   Solution: (1634)(2)(5) = (1634)

   (c) Determine if the permutation $\tau = (5321)$ is even or odd.
   Solution: Write τ as a product of transpositions: $(5321) = (51)(52)(53)$. Since τ is written as the product of three transpositions, τ is odd.

2. (a) List all the elements of the cyclic subgroup of the group $\mathbb{Z}_{14}$ under addition generated by 4.
   Solution: Using additive notation, $\langle 4 \rangle = \{4n \mid n \in \mathbb{Z}\}$ where $4n$ denotes $4 +_{14} 4 +_{14} \cdots_{14} 4$ (n terms). So, $\langle 4 \rangle = \{4, 8, 12, 2, 6, 10, 0\}$
   (b) What is the order of the element 4 in $\mathbb{Z}_{14}$?
   Solution: Since $|4| = |\langle 4 \rangle| = 7$ from part (a).
   (c) Does $\mathbb{Z}_{14}$ contain an element of order 5? If so, find one. If not, explain why not. (You may cite any theorems from class.)
   By Theorem 6.14, the order of any subgroup of $\mathbb{Z}_{14}$ must divide $|\mathbb{Z}_{14}| = 14$. Since the order of an element $a \in \mathbb{Z}_{14}$ is the order of the cyclic subgroup it generates, and $5 \nmid 14$, $\mathbb{Z}_{14}$ does not contain an element of order 5.

3. Let G be a group and suppose $a \in G$ generates a cyclic group of order 2 and is the *unique* such element. Show that $ax = xa$ for all $x \in G$. (Hint: Consider $(xax^{-1})^2$.)
   Solution: Suppose that $a \in G$ is the unique element that generates a subgroup of order two. Then $\langle a \rangle = \{a^n \mid n \in \mathbb{Z}\} = \{a, e\}$. In particular, this means $a^2 = e$. Now, let $x \in G$, and consider $(xax^{-1})^2$:

$$\begin{aligned} (xax^{-1})^2 &= (xax^{-1})(xax^{-1}) \\ &= xa(x^{-1}x)ax^{-1} && \text{(by associativity)} \\ &= xaeax^{-1} && \text{(by the definition of inverse)} \\ &= xa^2x^{-1} \\ &= xex^{-1} && \text{(since } a^2 = e\text{)} \\ &= xx^{-1} \\ &= e \end{aligned}$$

   So, we've shown

$$(xax^{-1})^2 = e.$$

   Now, recall that a is the unique element of order two. Hence, we must have $xax^{-1} = a$. Multiplying by x on the right, we get $xa = ax$.